Gaussian curvature conjecture for minimal graphs

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Some history of minimal surfaces

Minimal surface theory originates with Lagrange who in 1762 considered the variational problem of finding the surface $z = z(x, y)$ of least area stretched across a given closed contour. He derived the Euler–Lagrange equation for the solution

$$\frac{d}{dx}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)+\frac{d}{dy}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right)=0$$

He did not succeed in finding any solution beyond the plane. In 1776 Jean Baptiste Marie Meusnier discovered that the helicoid and catenoid satisfy the equation and that the differential expression corresponds to twice the mean curvature of the surface, concluding that surfaces with zero mean curvature are area-minimizing.

By expanding Lagrange's equation to

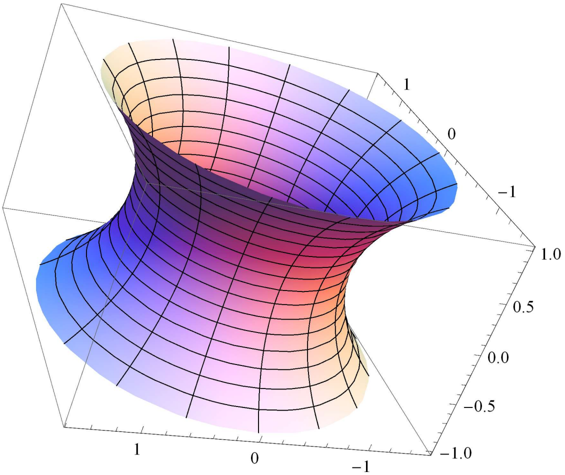
$$\left(1+z_x^2\right) z_{yy} - 2z_x z_y z_{xy} + \left(1+z_y^2\right) z_{xx} = 0$$

Gaspard Monge and Legendre in 1795 derived representation formulas for the solution surfaces. While these were successfully used by Heinrich Scherk in 1830 to derive his surfaces, they were generally regarded as practically unusable. Catalan proved in 1842/43 that the helicoid is the only ruled minimal surface (i.e. a surface generated by the motion of a straight line).

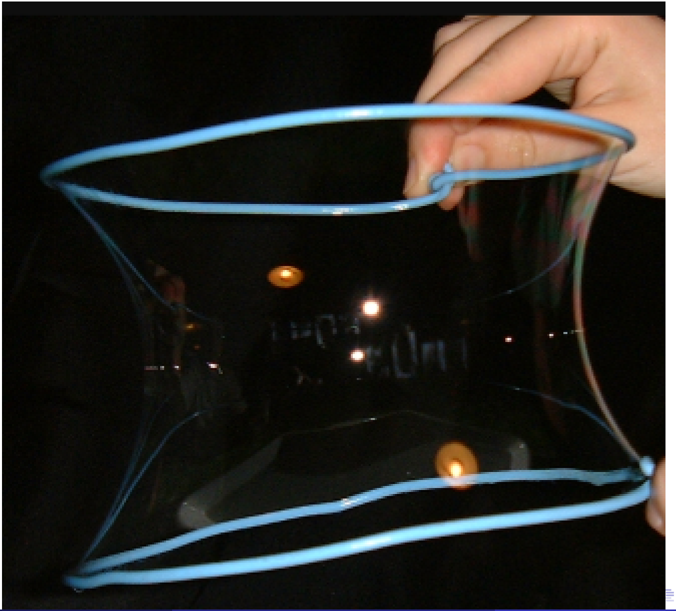
Progress had been fairly slow until the middle of the century when the Björling problem was solved using complex methods. The "first golden age" of minimal surfaces began. Schwarz found the solution of the Plateau problem for a regular quadrilateral in 1865 and for a general quadrilateral in 1867 (allowing the construction of his periodic surface families) using complex methods. Weierstrass and Enneper developed more useful representation formulas, firmly linking minimal surfaces to complex analysis and harmonic functions. Other important contributions came from Beltrami, Bonnet, Darboux, Lie, Riemann, Serret and Weingarten. Between 1925 and 1950 minimal surface theory revived, now mainly aimed at nonparametric minimal surfaces. The complete solution of the Plateau problem by Jesse Douglas and Tibor Radó was a major milestone. Bernstein's problem and Robert Osserman's work on complete minimal surfaces of finite total curvature were also important.

Another revival began in the 1980s. One cause was the discovery in 1982 by Celso Costa of a surface that disproved the conjecture that the plane, the catenoid, and the helicoid are the only complete embedded minimal surfaces in R3 of finite topological type. This not only stimulated new work on using the old parametric methods, but also demonstrated the importance of computer graphics to visualise the studied surfaces and numerical methods to solve the "period problem" (when using the conjugate surface method to determine surface patches that can be assembled into a larger symmetric surface, certain parameters need to be numerically matched to produce an embedded surface). Another cause was the verification by H. Karcher that the triply periodic minimal surfaces originally described empirically by Alan Schoen in 1970 actually exist. This has led to a rich menagerie of surface families and methods of deriving new surfaces from old, for example by adding handles or distorting them. Currently the theory of minimal surfaces has diversified to minimal submanifolds in other ambient geometries, becoming relevant to mathematical physics (e.g. the positive mass conjecture, the Penrose conjecture) and three-manifold geometry (e.g. the Smith conjecture, the Poincaré conjecture, the Thurston Geometrization Conjecture).

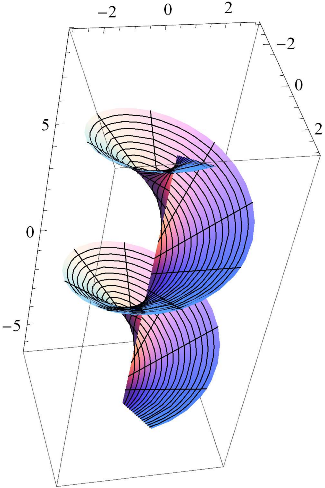
Catenoid



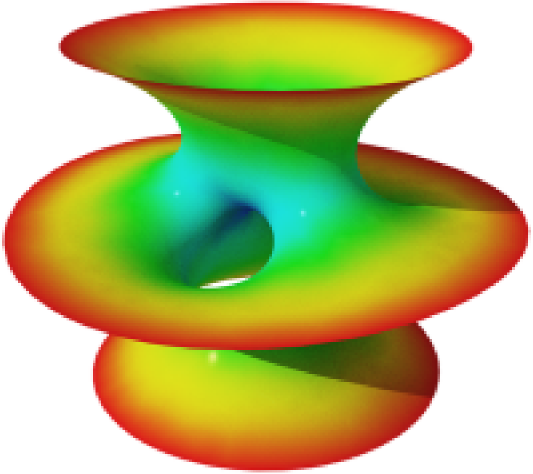
A soap film catenoid.



Helicoid



Costa minimal surface



Heinz-Hopf curvature problem

Let $D(w_0, R)$ be a disk in R^2 and let $\mathrm{f} : D(w_0, R) \to \mathrm{R}$ be a C^2 function that solves the minimal surface equation

$$\mathrm{f}_{uu}(1+\mathrm{f}_v^2) - 2\mathrm{f}_u\mathrm{f}_v\mathrm{f}_{uv} + \mathrm{f}_{vv}(1+\mathrm{f}_u^2) = 0.$$

The graph of f: $S = \mathrm{Graph}_{\mathrm{f}} = \{(u, v, \mathrm{f}(u, v))\}$ is called a minimal graph in R^3 over the disk $D(w_0, R)$. Then the Gaussian curvature of the graph S at a point $P = (u, v, \mathrm{f}(u, v))$ is given by

$$\mathcal{K}(P) = \frac{\mathrm{f}_{uu}\mathrm{f}_{vv} - \mathrm{f}_{uv}^2}{(1+\mathrm{f}_u^2+\mathrm{f}_v^2)^2}.$$

Observe that an important fact for minimal surfaces. Their mean curvature is equal to zero.

Assume that ξ is a point above w_0. A longstanding open problem in the theory of minimal surfaces is to determine the precise value of the constant c_0 in the inequality

$$|\mathcal{K}(\xi)| \leq \frac{c_0}{R^2}. \tag{1}$$

Proposed inequality

E. Hopf and E. Heinz (1951, 1952) found some numerical estimates for c_0, and it comes from their approach the following conjectured sharp inequality

$$|\mathcal{K}(\xi)| \leq \frac{\pi^2/2}{R^2}. \tag{2}$$

The inequality (2) has been also conjectured by Duren in his book *On harmonic mappings on the plane*.

Known results

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- J. C. C. Nitsche proved it for symmetric minimal surfaces in 1973.
- R. R. Hall gives the estimate of the curvature using some specific estimates (in the literature known as Heinz type estimates); we will describe it in more detail after introducing some basic notions.

Complex-analytic setting of the problem

Let $M \subset \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ be a minimal graph lying over the unit disc $\mathbb{D} \subset \mathbb{C}$.

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We denote by $f_z = \partial f / \partial z$ and $f_{\bar{z}} = \partial f / \partial \bar{z}$ the Wirtinger derivatives of f. The function ω defined by

$$\overline{f_{\bar{z}}} = \omega f_z \tag{3}$$

is called the *second Beltrami coefficient* of f, and the above equation is the *second Beltrami equation* with f as a solution. Observe that $\bar{f}_z = \overline{f_{\bar{z}}}$ and this notation will be used in the sequel.

Orientation preserving of f is equivalent to $\mathrm{Jac}(f) = |f_z|^2 - |f_{\bar{z}}|^2 > 0$, hence to $|\omega| < 1$ on $\mathbb{D}$. Furthermore, the function ω is holomorphic whenever f is harmonic and orientation preserving. (In general, it is meromorphic when f is harmonic.) To see this, let

$$u + \imath v = f = h + \overline{g} \tag{4}$$

be the canonical decomposition of the harmonic map $f : \mathbb{D} \to \mathbb{D}$, where h and g are holomorphic functions on the disk.

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be the canonical decomposition of the harmonic map $f : \mathbb{D} \to \mathbb{D}$, where h and g are holomorphic functions on the disk. Then,

$$f_z = h', \quad f_{\bar{z}} = \overline{g}_{\bar{z}} = \overline{g'}, \quad \omega = \overline{f_{\bar{z}}}/f_z = g'/h'. \tag{5}$$

In particular, the second Beltrami coefficient ω equals the meromorphic function g'/h' on $\mathbb{D}$. In our case we have $|\omega| < 1$, so it is a holomorphic map $\omega : \mathbb{D} \to \mathbb{D}$.

We now consider the Enneper–Weierstrass representation of the minimal graph $\varpi = (u, v, T) : \mathbb{D} \to M \subset \mathbb{C} \times \mathbb{R}$ over f. We have

$$\begin{aligned} u(z) &= \Re f(z) = \Re \int_0^z \phi_1(\zeta) d\zeta \\ v(z) &= \Im f(z) = \Re \int_0^z \phi_2(\zeta) d\zeta \\ T(z) &= \Re \int_0^z \phi_3(\zeta) d\zeta \end{aligned}$$

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where

$$\begin{aligned} \phi_1 &= 2(u)_z = 2(\Re f)_z = (h + \bar{g} + \bar{h} + g)_z = h' + g', \\ \phi_2 &= 2(v)_z = 2(\Im f)_z = \imath(\bar{h} + g - h - \bar{g})_z = \imath(g' - h'), \\ \phi_3 &= 2(T)_z = \sqrt{-\phi_1^2 - \phi_2^2} = \pm 2\imath\sqrt{h'g'}. \end{aligned}$$

Let us introduce the notation $\mathsf{p} = f_z$. We have

$$\mathsf{p} = f_z = (\Re f)_z + \imath(\Im f)_z = \frac{1}{2}(h' + g' + h' - g') = h'. \tag{6}$$

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By using $\omega = \overline{f_{\bar{z}}}/f_z = g'/h'$ (see (5)), it follows that

$$\phi_1 = h' + g' = \mathsf{p}(1+\omega), \quad \phi_2 = -\imath(h' - g') = -\imath\mathsf{p}(1-\omega), \quad \phi_3 = \pm 2\imath\mathsf{p}\sqrt{\omega}.$$

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From the formula for ϕ_3 we infer that ω has a well-defined holomorphic square root:

$$\omega = \mathrm{q}^2, \qquad \mathrm{q} : \mathbb{D} \to \mathbb{D} \ \text{ holomorphic.} \tag{7}$$

In terms of the Enneper–Weierstrass parameters (p, q) given by (6) and (7) we obtain

$$\phi_1 = \mathsf{p}(1+\mathsf{q}^2), \quad \phi_2 = -\imath \mathsf{p}(1-\mathsf{q}^2), \quad \phi_3 = -2\imath \mathsf{p}\mathsf{q}. \tag{8}$$

(The choice of sign in ϕ_3 is a matter of convenience; since we have two choices of sign for q in (7), this does not cause any loss of generality.) Hence,

$$\varpi(z) = \left(\Re f(z), \Im f(z), \Im \int_0^z 2\mathsf{p}(\zeta)\mathsf{q}(\zeta)d\zeta\right), \quad z \in \mathbb{D}.$$

The curvature $\mathcal{K}$ of the minimal graph M is expressed in terms of (h, g, ω) (5), and in terms of the Enneper–Weierstrass parameters (p, q), by

$$\mathcal{K} = -\frac{|\omega'|^2}{|h'g'|(1+|\omega|)^4} = -\frac{4|\mathrm{q}'|^2}{|\mathrm{p}|^2(1+|\mathrm{q}|^2)^4}, \tag{9}$$

where $\mathrm{p} = f_z$ and $\omega = \mathrm{q}^2 = \overline{f_{\bar{z}}}/f_z$.

An approach - Heinz type inequality

In the last formula, using Schwarz lemma we get:

$$\begin{aligned}|\mathcal{K}| &\le \frac{4(1-|\mathrm{q}(0)|^2)^2}{|\mathrm{p}(0)|^2(1+|\mathrm{q}(0)|^2)^4} \\ &= \frac{4(1-|\omega(0)|)^2}{(|h'(0)|^2+|g'(0)|^2)(1+|\omega(0)|)^2} \\ &\le \frac{4}{|h'(0)|^2+|g'(0)|^2},\end{aligned}$$

We pose the following open problem: Whether this inequality $|h'(0)|^2+|g'(0)|^2 \ge \frac{8}{\pi^2}$ for for a harmonic mapping $f = h + \overline{g} : \mathbb{D} \to \mathbb{D}$ which is onto, $f(0) = 0$ and $\frac{g'}{h'} = \mathrm{q}^2$ i.e. the dilation is the square of an analytic function? This would solve the Hopf-Heinz problem. We will another strategy instead. We will use equation (9) directly.

It turns out that this question is very hard and it remains open. However, this approach still gives some satisfactory results.

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The example of Scherk surface gives and the appropriate h and g (the dilation is equal to z^2) shows that in the estimate $\mathcal{K} \le c$ the constant c cannot be smaller than conjectured $\frac{\pi^2}{2}$.

The conjectured "nearly extremal" minimal surface gives the motivation for the following:

Definition

Assume that $Q = Q(a, b, c, d)$ is a bicentric quadrilateral inscribed in the unit disk $\mathbb{D}$. A minimal graph $S = \{(u, v, \mathrm{f}(u, v)), (u, v) \in Q\}$ over the quadrilateral Q is called a Scherk type surface if it satisfies $\mathrm{f}(u, v) \to +\infty$ when $(u, v) \to \zeta \in (a, b) \cup (c, d)$ and $\mathrm{f}(u, v) \to -\infty$ when $(u, v) \to \zeta \in (b, c) \cup (a, d)$.

Comparison theorem

For every $w \in \mathbb{D}$, there exist four different points $a_0, a_1, a_2, a_3 \in \mathbf{T}$ and a harmonic mapping f of the unit disk onto the bicentric quadrilateral $Q(a_0, a_1, a_2, a_3)$ that solves the Beltrami equation

$$\bar{f}_z(z) = \left(\frac{w + \frac{\imath(1-w^4)z}{|1-w^4|}}{1 + \frac{\imath\overline{w}(1-w^4)z}{|1-w^4|}} \right)^2 f_z(z), \tag{10}$$

$|z| < 1$ and satisfies the initial condition $f(0) = 0$, $f_z(0) > 0$.

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$|z| < 1$ and satisfies the initial condition $f(0) = 0$, $f_z(0) > 0$.
The mapping f gives rise to a Scherk type minimal surface $S^\diamond : \zeta = \mathrm{f}^\diamond(u, v)$ over the quadrilateral $Q(a_0, a_1, a_2, a_3)$, containing the point $\xi = (0, 0, 0)$ above the origin so that its Gaussian normal is

$$\mathrm{n}^\diamond_\xi = -\frac{1}{1 + |w|^2}(2\Im w, 2\Re w, -1 + |w|^2),$$

and such that $D_{uv}\mathrm{f}^\diamond(0, 0) = 0$.

Moreover, every other non-parametric minimal surface $S : \zeta = \mathrm{f}(u, v)$ over the unit disk, containing the point ξ above zero, with $\mathrm{n}_\xi = \mathrm{n}_\xi^\diamond$ and $D_{uv}\mathrm{f}(0,0) = 0$ satisfies the sharp inequality

$$|\mathcal{K}_S(\xi)| < |\mathcal{K}_{S^\diamond}(\xi)|.$$

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This Theorem (proved in [Kalaj]) is the first step in proving the main resul. So it remains describe all Scherk type minimal surfaces above the unit disk and find the supremum of its Gaussian curvatures in the point above the center of the disk!

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In the following slides we will describe describe all Scherk type minimal surfaces. and using it we will prove the desired sharp inequality for curvature.

Construction of two-parameter family of Scherk type surfaces

Let p, q and β satisfies

$$0 \le p \le q \le q - p + \pi \le 2\pi \tag{11}$$

and

$$p \le q - p + 2\beta \le 2\pi - p \le 2\pi - q + p + 2\beta \le 2\pi + p. \tag{12}$$

By Sheil-Small theorem, harmonic extension f_1 of the function

$$F_1(\psi) = \begin{cases} a_1 = e^{\imath t_1}, & \text{if } \psi \in [0, s_1); \\ a_2 = e^{\imath t_2}, & \text{if } \psi \in [s_1, s_2); \\ a_3 = e^{\imath t_3}, & \text{if } \psi \in [s_2, s_3); \\ a_4 = e^{\imath t_4}, & \text{if } \psi \in [s_3, 2\pi). \end{cases}$$

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with $s_1 = p$, $s_2 = \pi$, $s_3 = \pi + q - p$, $t_1 = \frac{q+\pi}{2} + \beta - p$, $t_2 = \frac{5\pi - q}{2} - p - \beta$, $t_3 = \frac{5\pi - 3q}{2} + \beta + p$ and $t_4 = \frac{5\pi - q}{2} + p - \beta$ solves Beltrami equation

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After the rotation R of the quadrilateral in co-domain for the angle $\beta + \frac{q-\pi}{2}$, we get the quadrilateral with vertices $e^{\imath x}, e^{\imath y}, e^{\imath s}, e^{\imath p}$ where $x = q - p + 2\beta, y = 2\pi - p, s = 2\pi - q + p + 2\beta$, and $\tan\beta = \frac{\sin\frac{q}{2}}{\cos(\frac{q}{2} - p)}$.
Last identity is equivalent to the condition that the quadrilateral with vertices $e^{\imath x}, e^{\imath y}, e^{\imath s}, e^{\imath p}$ where
$x = q - p + 2\beta, y = 2\pi - p, s = 2\pi - q + p + 2\beta$ is bicentric.

Using Mobius transform we can get a new f which maps the unit disk onto a bicentric quadrilateral whose vertices are $e^{\imath p}$, $e^{\imath x}$, $e^{\imath y}$, and $e^{\imath s}$ and such that the limit points for the mapping f are $1, e^{\imath\alpha}, -1, -e^{\imath\alpha}$ (i.e. makes a rectangle). More precisely,

$$f(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(t-\psi)} F(e^{\imath\psi})d\psi$$

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$$F(e^{\imath\psi}) = \begin{cases} e^{\imath x}, & \text{if } \psi \in [0,\alpha); \\ e^{\imath y}, & \text{if } \psi \in [\alpha,\pi); \\ e^{\imath s}, & \text{if } \psi \in [\pi,\pi+\alpha); \\ e^{\imath p}, & \text{if } \psi \in [\pi+\alpha,2\pi). \end{cases}$$

Here $\cos\alpha = \frac{\sin(q-p)-\sin p}{\sin(q-p)+\sin p}$!

We again infer the existence of $a \in \mathbb{D}, b \in \mathbb{C}, \theta \in [0, 2\pi]$ such that

$$\mathrm{p} = \frac{b(1 - z\overline{a})^2}{(z^2 - 1)(z^2 - e^{2\imath\alpha})} \tag{14}$$

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and

$$\mathrm{q} = e^{\imath\theta} \frac{z - a}{1 - z\bar{a}}, \tag{15}$$

where $\mathrm{p} = h'$ and $\mathrm{pq}^2 = g'$. Note that a is the zero of the Möbius transformation M.

We again infer the existence of $a \in \mathbb{D}, b \in \mathbb{C}, \theta \in [0, 2\pi]$ such that

$$\mathrm{p} = \frac{b(1 - z\bar{a})^2}{(z^2 - 1)(z^2 - e^{2\imath\alpha})} \tag{14}$$

and

$$\mathrm{q} = e^{\imath\theta} \frac{z - a}{1 - z\bar{a}}, \tag{15}$$

where $\mathrm{p} = h'$ and $\mathrm{pq}^2 = g'$. Note that a is the zero of the Möbius transformation M.

After some computation, we get a, b, θ, also, there is a more appropriate form of f.

First, for further calculations, we express f as

$$f(z) = u(z) + \imath v(z) + f(0), \tag{16}$$

with

$$\begin{aligned}
f(0) &= \frac{\alpha e^{2\imath\beta}\cos(p-q) + (\pi-\alpha)\cos p}{\pi}, \\
u(re^{\imath t}) &= \frac{\cos(q-p+2\beta) - \cos p}{\pi}\tan^{-1}\frac{r\sin(\alpha-t)}{1-r\cos(\alpha-t)} \\
&+ \frac{\cos p - \cos(p-q+2\beta)}{\pi}\tan^{-1}\frac{r\sin t}{1+r\cos t} \\
&+ \frac{\cos p - \cos(p-q+2\beta)}{\pi}\tan^{-1}\frac{r\sin(\alpha-t)}{1+r\cos(\alpha-t)} \\
&+ \frac{\cos(q-p+2\beta) - \cos p}{\pi}\tan^{-1}\frac{r\sin t}{1-r\cos t}
\end{aligned} \tag{17}$$

and

$$v(re^{\imath t}) = \frac{\sin(q-p+2\beta)+\sin p}{\pi}\tan^{-1}\frac{r\sin(\alpha-t)}{1-r\cos(\alpha-t)}$$
$$-\frac{\sin p+\sin(p-q+2\beta)}{\pi}\tan^{-1}\frac{r\sin t}{1+r\cos t}$$
$$+\frac{\sin p-\sin(p-q+2\beta)}{\pi}\tan^{-1}\frac{r\sin(\alpha-t)}{1+r\cos(\alpha-t)}$$
$$+\frac{\sin(q-p+2\beta-\sin p)}{\pi}\tan^{-1}\frac{r\sin t}{1-r\cos t}, \quad (18)$$

and

$$
\begin{aligned}
v(re^{\imath t}) = {} & \frac{\sin(q-p+2\beta)+\sin p}{\pi}\tan^{-1}\frac{r\sin(\alpha-t)}{1-r\cos(\alpha-t)}\\
& -\frac{\sin p+\sin(p-q+2\beta)}{\pi}\tan^{-1}\frac{r\sin t}{1+r\cos t}\\
& +\frac{\sin p-\sin(p-q+2\beta)}{\pi}\tan^{-1}\frac{r\sin(\alpha-t)}{1+r\cos(\alpha-t)}\\
& +\frac{\sin(q-p+2\beta-\sin p)}{\pi}\tan^{-1}\frac{r\sin t}{1-r\cos t},
\end{aligned}
\tag{18}
$$

while

$$
a=\frac{\cos(q-p)-\cos p-\imath\sin q}{1-\cos q+2\sqrt{\sin p\sin(q-p)}+\imath(\sin(q-p)-\sin p)}. \tag{19}
$$

and

$$v(re^{\imath t}) = \frac{\sin(q-p+2\beta)+\sin p}{\pi}\tan^{-1}\frac{r\sin(\alpha-t)}{1-r\cos(\alpha-t)}$$
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while

$$a = \frac{\cos(q-p)-\cos p-\imath\sin q}{1-\cos q+2\sqrt{\sin p\sin(q-p)}+\imath(\sin(q-p)-\sin p)}. \tag{19}$$

Or,

$$a = \sqrt{\frac{1 - \sqrt{\sin p \sin(q-p)}}{1 + \sqrt{\sin p \sin(q-p)}}}(\cos\delta + \imath \sin\delta), \tag{20}$$

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where

$$\cos\delta=-\frac{\sin(\frac{q}{2}-p)}{\sin\frac{q}{2}\sqrt{1-\sin(q-p)\sin p}}$$

and

$$\sin\delta=-\frac{\cos\frac{q}{2}\sqrt{\sin(q-p)\sin p}}{\sin\frac{q}{2}\sqrt{1-\sin(q-p)\sin p}}.$$

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$$\sin\delta=-\frac{\cos\frac{q}{2}\sqrt{\sin(q-p)\sin p}}{\sin\frac{q}{2}\sqrt{1-\sin(q-p)\sin p}}.$$

Also,

$$|b|^2=\frac{2}{\pi^2}\left(2\sin^2(p-q)\sin^2\frac{\alpha}{2}+2\sin^2 p\cos^2\frac{\alpha}{2}-2\sin p\sin(p-q)\sin(2\beta)\sin\alpha\right) \tag{21}$$

and

Or,

$$a=\sqrt{\frac{1-\sqrt{\sin p\sin(q-p)}}{1+\sqrt{\sin p\sin(q-p)}}}(\cos\delta+\imath\sin\delta), \tag{20}$$

where

$$\cos\delta=-\frac{\sin(\frac{q}{2}-p)}{\sin\frac{q}{2}\sqrt{1-\sin(q-p)\sin p}}$$

and

$$\sin\delta=-\frac{\cos\frac{q}{2}\sqrt{\sin(q-p)\sin p}}{\sin\frac{q}{2}\sqrt{1-\sin(q-p)\sin p}}.$$

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and

$$\theta=\tan^{-1}[\cos(p-q)\tan(p)].$$

Finally, inserting these values, we get that the Gaussian curvature at the point over the $0 = f(z_\circ)$ is equal to

$$\mathcal{K} = -\frac{4(1-|a|^2)^2}{|b|^2}\frac{|z_\circ^2-1|^2|z_\circ^2-e^{2\imath\alpha}|^2}{(|1-z_\circ\overline{a}|^2+|z_\circ-a|^2)^4}. \tag{22}$$

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Or

$$\begin{aligned}-\mathcal{K} &= \frac{4\pi^2(1+\sin p\sin(q-p))}{\left(1+\sqrt{\sin p\sin(q-p)}\right)^4}\frac{|1-z_\circ^2|^2|1-z_\circ^2e^{2\imath\alpha}|^2}{((1+|z_\circ|^2)(1+|a|^2)-4\Re(a\bar{z}_\circ))^4}\\ &= \frac{\pi^2}{4}(1+\sin p\sin(q-p))\frac{|1-z_\circ^2|^2|1-z_\circ^2e^{2\imath\alpha}|^2}{((1+|z_\circ|^2)-4\frac{\Re(a\bar{z}_\circ)}{(1+|a|^2)})^4}.\end{aligned} \tag{23}$$

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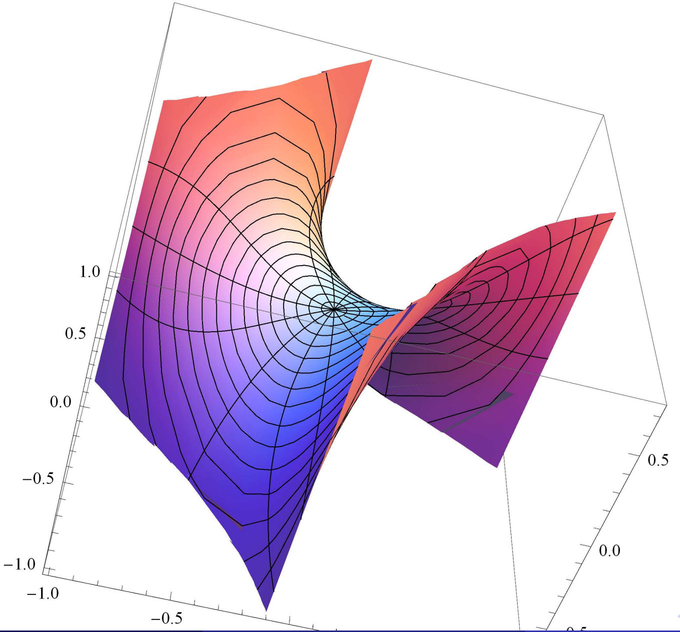
It is not so beautiful, but $\Re(a\bar{z}_\circ)\le 0$ will help very much! And it really holds!

Based on (17), (18) we obtain a Scherk type surface $\varpi(z) = (u(z), v(z), T(z))$ in Figure 4


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.5
0.0

over the quadrilateral shown in Figure 5

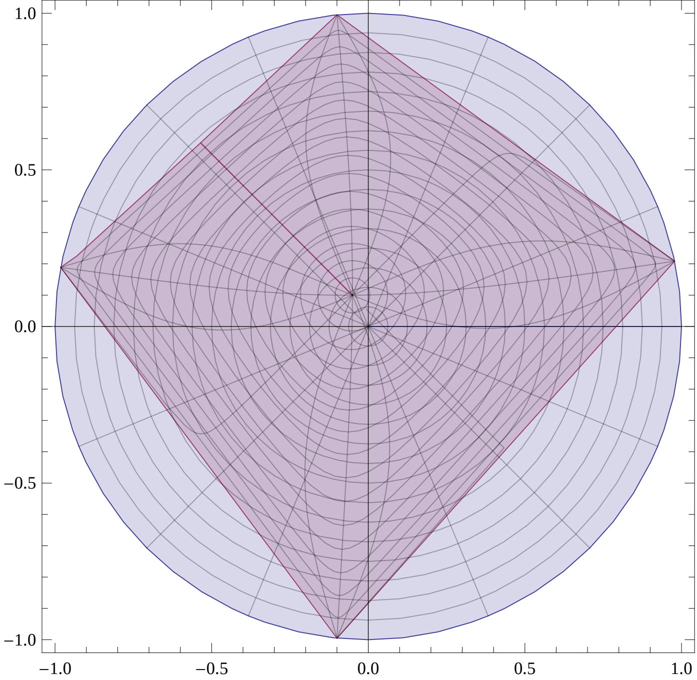

1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

Figure: A bicentric quadrilateral inscribed in the unit disk for $p = \pi/2 + 0.1$, $q = \pi - 0.1$.

Localization lemma and proof of conjecture

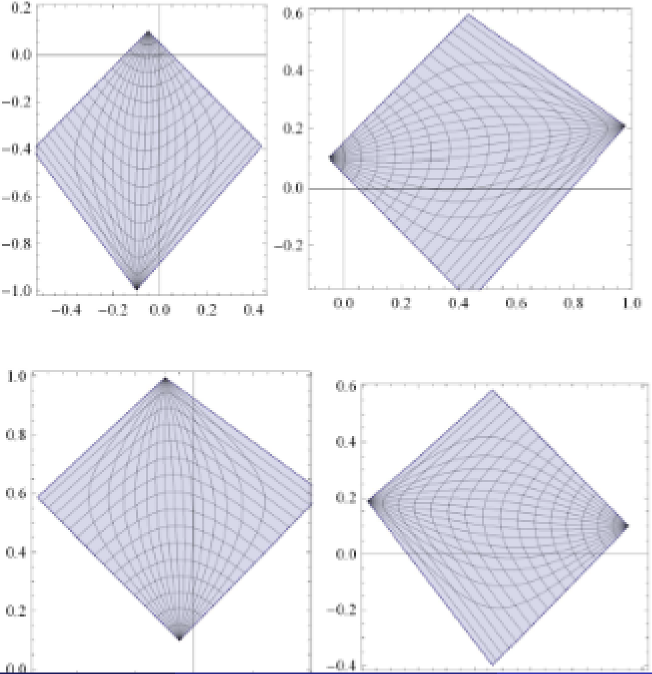
Lemma (Localization of the zero $z_\circ$)

Assume that (p, q) is a pair of real numbers that belongs to the domain $\mathcal{R}$ and assume that $z_\circ$ is the zero of f. Then we have

a) $\arg z_\circ \in (\alpha, \pi)$, *for* $q < \pi, q < 2p$,
b) $\arg z_\circ \in (\pi + \alpha, 2\pi)$, *for* $q > \pi, q > 2p$,
c) $\arg z_\circ \in (\pi, \pi + \alpha)$, *for* $q \in (\pi, 2p)$,
d) $\arg z_\circ \in (0, \alpha)$, *for* $q \in (2p, \pi)$.

This is a kind of harmonic argument principle. From this we have $\Re(a\bar{z}_\circ) \leq 0$ and, then, from the formula (22) we get

$$|\mathcal{K}| \leq \frac{\pi^2}{2} \frac{|1 - z_\circ^2|^2 |z_\circ^2 - e^{2\imath\alpha}|^2}{(1 + |z_\circ|^2)^4} \leq \frac{\pi^2}{2},$$


0.2 0.0 −0.2 −0.4 −0.6 −0.8 −1.0
−0.4 −0.2 0.0 0.2 0.4
0.6 0.4 0.2 0.0 −0.2
0.0 0.2 0.4 0.6 0.8 1.0
1.0 0.8 0.6 0.4 0.2 0.0
0.6 0.4 0.2 0.0 −0.2 −0.4

The images of four consecutive sectors, $\arg z \in [\alpha, \pi]$, $[\pi, \pi + \alpha]$, $[\pi + \alpha, 2\pi]$ and $[0, \alpha]$ under the harmonic mapping f. In this case $p = \pi/2 + 0.1$, $q = \pi - 0.1$, so $q < \min\{2p, \pi\}$. Notice that the first curvilinear quadrilateral contains the zero, and we are in the case a).

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Faleminderit!